

Maneuver and Vibration Control of Hybrid Coordinate Systems Using Lyapunov Stability Theory

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In this study, we present a generalized control law design methodology that covers a large class of systems, especially flexible structures described by hybrid discrete/distributed coordinate systems. The Lyapunov stability theory is used to develop globally stabilizing control laws. A hybrid version of Hamilton's canonical equations is introduced, which provides a direct path to designing stabilizing control laws for general nonlinear hybrid systems. The usual loss of robustness due to model reduction is overcome by this Lyapunov approach, which does not require truncation of the flexible systems into finite dimensional discrete systems.

I. Introduction

DURING the past few decades, control problems¹⁻⁵ for flexible space structures have received significant attention. The main difficulty of flexible space structures is due to the flexibility inherent in the structures that are infinite dimensional systems. Since the equations of motion of infinite dimensional systems are usually described by partial differential equations (PDE), some approximations are necessary to develop finite dimensional systems to be used for conventional control law design, such as closed-loop state feedback^{6,7} and open-loop control⁸⁻¹⁰ techniques, as well as for simulation purposes. The limited dimension of practical controller design often requires discretization of the original PDE model into a system of finite dimensional ordinary differential equations (ODE), and therefore truncation errors are always introduced when the truncated models are used. In particular, for large flexible space structures (LFSS), the model reduction issue has been a major consideration in control law design due to the high dimensionality of the original model. The design of robust control laws to accommodate modeling error such as spill-over effect due to unmodeled dynamics and uncertain system parameters is still under investigation.

The derivation of a finite dimensional ODE model from an original infinite dimensional PDE model has direct effects on the performance of the controller. Hence, the control law design suffers from inherent limitations on the performance and, especially, stability guarantees due to whatever model error is present in a particular application. In some recent approaches,¹¹⁻¹⁵ a control law design approach based on Lyapunov stability theory for developing stabilizing control laws for flexible space structures has been investigated. The Lyapunov approach¹¹⁻¹⁵ uses weighted system energy of the original PDE model and finds a control law that decreases system energy so that the system is stabilized toward a desired equilibrium point. The stability guarantees of this approach are robust to discretization errors in the sense that we can design stabilizing control laws directly using the original PDE model without relying on spatial discretization. On the other hand,

finite dimensional approximations are still introduced to tune the gains over the stable region. In fact, the Lyapunov approach has been frequently used for rigid spacecraft control, and it is only in the recent literature that the Lyapunov approach has been extended to design control laws for the maneuver and vibration control of flexible space structures.

In this paper, we develop Lyapunov control laws for the maneuver and vibration control of flexible space structures. Generalizations of the Lyapunov approach are made by introducing Hamilton's canonical equations for the flexible systems described by a coupled set of ordinary, partial, and integral equations. The control laws developed will be in output feedback forms, including feedback on internal boundary forces. The boundary force feedback is important for a multibody system where each substructure is connected with adjacent substructures, if the corresponding boundary forces can be measured using load cells or strain gauges.

II. Discrete System Case

For a general n -dimensional mechanical discrete system, the governing equations of motion are expressed as

$$M(q)\ddot{q} + \Gamma(q, \dot{q}) + K(q) = Q \quad (1)$$

where $q = [q_1, q_2, \dots, q_n]^T$ is the vector of generalized coordinate, $\Gamma(q, \dot{q})$ denotes gyroscopic coupling effects, $M(q)$ is a mass matrix, $K(q)$ denotes the vector of potential and centrifugal forces, and $Q = [Q_1, Q_2, \dots, Q_n]$ denotes the nonconservative external force vector. The Hamiltonian corresponding to the previous system is defined to be¹⁶

$$H(p, q) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}) \quad (2)$$

where the generalized momentum p_i and the system Lagrangian L are defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \text{where} \quad L = T(q, \dot{q}) - V(q) \quad (3)$$

Then the associated Hamilton's canonical equations are obtained as follows:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} + Q_i, \quad i = 1, 2, \dots, n \quad (4)$$

where $(\dot{}) \equiv d/dt()$. Although the Hamiltonian can be expressed as a function of either $\{q_i, \dot{q}_i\}$ or $\{q_i, p_i\}$, this latter set of coordinates is implicit in the partial derivatives of Eq. (4). The Hamiltonian of the system has a system-specific structure depending on the system under consideration. In

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particular, for a natural system where the kinetic energy is represented as a symmetric quadratic function of the velocity coordinate of the system, the Hamiltonian is a measure of total system energy. Since most mechanical systems are natural systems, the Hamiltonian reduces to the total system energy ($H = T + V$). This motivates the use of a Hamiltonian as a Lyapunov function candidate. Hence, to find stabilizing control laws by a Lyapunov method, we choose a candidate Lyapunov function U as the system Hamiltonian

$$U = H(p, q) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}) \quad (5)$$

Note that the generalized velocity \dot{q} must be eliminated in $L(q, \dot{q})$ as a function of (q, p) , using Eq. (3) to obtain $U(q, p)$ in Eq. (5). According to the Lyapunov stability theory, we investigate the time rate of change of the candidate Lyapunov function. Hamilton's canonical equations in Eq. (4) are used to evaluate the derivative of the Lyapunov function¹⁶

$$\frac{dU}{dt} = \frac{dH(p, q)}{dt} = \sum_{i=1}^n \frac{\partial H}{\partial p_i} \dot{p}_i \quad (6)$$

and so the simplest family of stabilizing control laws requiring $dU/dt < 0$ is obtained in the form of linear feedback

$$\dot{Q}_i = -k_i \frac{\partial H}{\partial p_i} = -k_i \dot{q}_i, \quad k_i > 0 \quad i = 1, 2, \dots, n$$

It can be easily seen that

$$\dot{q}_i = \sum_{j=1}^n f_{ij}(q) p_j, \quad \text{or} \quad \dot{q} = F(q)p, \quad \frac{dU}{dt} = -p^T F^T \Lambda F p$$

where Λ is a diagonal matrix consisting of k_i . Assuming U of Eq. (5) is indeed positive definite with its global minimum at the origin, then the control laws are globally stabilizing in the sense of Lyapunov stability criteria. Since Eq. (6) with the previous control law becomes $dU/dt = -\sum_{i=1}^n k_i \dot{q}_i^2$ not containing q_i (or p_i), we must verify (for nonlinear systems) that the p_i cannot vanish identically for nonzero q_i . In essence, we require for asymptotic stability that the origin is the *only equilibrium* point of the closed-loop system. It is noteworthy that the control laws are robust since all of the nonlinearities are absorbed into the definition of the Lyapunov function, which is the Hamiltonian of the system. One simple yet often subtle point is this: this law does not explicitly depend on the system model assumptions; therefore *all* controllable systems (having some actual but unknown U that is positive definite with its global minimum at the origin) are stabilized by the same control law! Note dU/dt is the instantaneous work rate, which is a *kinematic* quantity. Now we seek to generalize this idea into hybrid coordinate systems so that analogous robustness and global stability properties of the control laws for flexible systems can be established.

III. Generalization into Hybrid Coordinate Systems

To design stabilizing control laws for hybrid coordinate systems we need a generalization of Hamilton's classical canonical equations. The hybrid version of Hamilton's canonical equations is not as explicit or compactly written as the equations of the discrete version, however. For complicated systems, these equations can be best developed by dividing the structure into a collection of distributed and discrete coordinate substructures. Before introducing the generalized Hamilton's canonical equations, we introduce a hybrid version of Lagrange's equations¹⁷ that is needed for deriving the hybrid version of Hamilton's canonical equations.

A. Hybrid Version of Hamilton's Canonical Equations

First, we consider a hybrid coordinate dynamical system and assume that the Lagrangian $L = T - V$, in which T is the

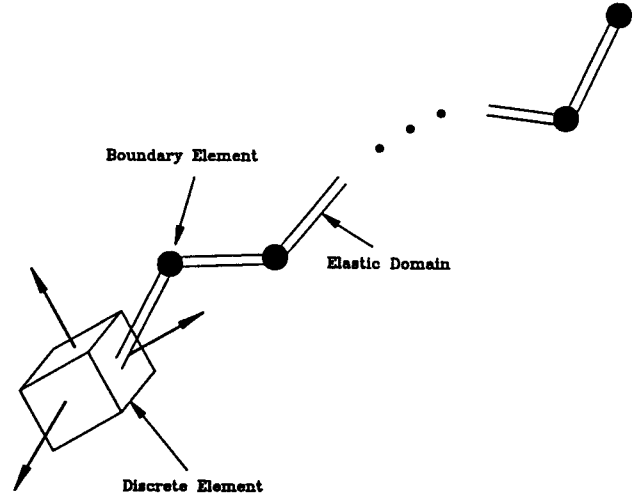


Fig. 1 Multiply connected flexible space structure.

kinetic energy and V is the potential energy, can be written in the general form $L = L(t, P, q_i, \dot{q}_i, w_j, \dot{w}_j, w'_j, \dot{w}'_j)$, where $q_i = q_i(t)$ ($i = 1, 2, \dots, m$) are generalized coordinates describing rigid-body motions and other discrete coordinate dynamics of the hybrid system, and $w_j = w_j(P, t)$ ($j = 1, 2, \dots, n$) are distributed coordinates describing elastic motions relative to the rigid-body motions of an undeformed body-fixed spatial position P . However, general motions and deformation states are allowed, including in-plane, out-of-plane, torsional, and axial deformations. For notational simplicity, we will use $w_j(P, t) \equiv w_j(P)$. Also, we use $q = [q_1, q_2, \dots, q_m]^T$ and $w = [w_1, w_2, \dots, w_n]^T$ for the representation of vectors of discrete and distributed coordinate configuration vectors.

Now for a system that consists of n multiply connected elastic domains as in Fig. 1, the definition of a generic distributed coordinate vector w_i ($i = 1, 2, \dots, n$) can be introduced for the i th elastic domain.

For convenience in the present discussion, we consider each elastic body to have a beamlike or rodlike structure, with only one elastic independent coordinate (x_i) for the i th body. For notational convenience, we define $\underline{w}(l)$ as $\underline{w}(l, t) = \{w_1(l_1), w_2(l_2), \dots, w_n(l_n)\}^T$, and $\dot{\underline{w}}(l)$, $\underline{w}'(l)$, and $\dot{\underline{w}}'(l)$ are defined in a similar manner. Then the Lagrangian of the system can be written in the form¹⁷

$$L = L_D(q, \dot{q}) + \sum_{i=1}^n \int_{l_{0i}}^{l_i} \tilde{L}^i[w_i, \dot{w}_i, w'_i, \dot{w}'_i, q, \dot{q}, \underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l)] dx_i + L_B[\underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l), q, \dot{q}] \quad (7)$$

where $(\dot{}) \equiv \partial/\partial t()$ denotes time derivative of () , regarding () as a function of x_i and t , and $(\cdot)' \equiv \partial/\partial x(\cdot)$. In addition,

$$L_D = T_D(q, \dot{q}) - V_D(q, \dot{q})$$

$$\begin{aligned} \tilde{L}^i &= \hat{T}^i[w_i, \dot{w}_i, w'_i, \dot{w}'_i, w''_i, q, \dot{q}, \underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l)] \\ &\quad - \hat{V}^i[w_i, \dot{w}_i, w'_i, \dot{w}'_i, w''_i, q, \dot{q}, \underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l)] \\ L_B &= T_B[\underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l), q, \dot{q}] \\ &\quad - V_B[\underline{w}(l), \dot{\underline{w}}(l), \underline{w}'(l), \dot{\underline{w}}'(l), q, \dot{q}] \end{aligned}$$

where $L_D(q, \dot{q})$ is a Lagrangian contribution from discrete coordinate systems (q) , and \tilde{L}^i is a Lagrangian density function from i th continuous element. In addition, L_B is the Lagrangian contribution from combinations of boundary elements whose characteristics are generally coupled and non-

linear for multiple elastic domain systems. Moreover, the non-conservative virtual work of the hybrid coordinate system can be represented by

$$\delta W_{nc} = \mathbf{Q}^T \mathbf{q} + \sum_{i=1}^n \left[\int_{l_{0i}}^{l_i} \mathbf{f}^{iT} \delta \mathbf{w} \, dx_i + \mathbf{f}_1^{iT} \delta \mathbf{w}_i(l_i) + \mathbf{f}_2^{iT} \delta \mathbf{w}_i'(l_i) \right] \quad (8)$$

where \mathbf{f}^i is a distributed force vector, and \mathbf{f}_1^i and \mathbf{f}_2^i are point force vectors applied at the boundary elements of the corresponding elastic domains. Based on the definition of the Lagrangian and virtual work expression, the hybrid version of Lagrange's equations, including discrete boundary elements, are obtained as follows¹⁷:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q} \quad (9a)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i} \right) - \frac{\partial \hat{L}^i}{\partial \mathbf{w}_i} + \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'} \right) - \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i''} \right) \\ - \frac{\partial^2}{\partial x_i \partial t} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'} \right) = \mathbf{f}^i \end{aligned} \quad (9b)$$

and the boundary conditions are

$$\begin{aligned} \left[-\frac{d}{dt} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'} \right) - \frac{\partial \hat{L}^i}{\partial \mathbf{w}_i'} - \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i''} \right) \right] \delta \mathbf{w}_i \Big|_{l_{0i}}^{l_i} \\ + \left\{ \frac{\partial \hat{L}_B}{\partial \mathbf{w}_i(l_i)} - \frac{d}{dt} \left[\frac{\partial \hat{L}_B}{\partial \dot{\mathbf{w}}_i(l_i)} \right] \right\} \delta \mathbf{w}_i(l_i) + \mathbf{f}_1^{iT} \delta \mathbf{w}_i(l_i) = 0 \\ \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i''} \delta \mathbf{w}_i'' \Big|_{l_{0i}}^{l_i} + \left\{ \frac{\partial \hat{L}_B}{\partial \dot{\mathbf{w}}_i'(l_i)} - \frac{d}{dt} \left[\frac{\partial \hat{L}_B}{\partial \dot{\mathbf{w}}_i'(l_i)} \right] \right\} \delta \mathbf{w}_i'(l_i) \\ + \mathbf{f}_2^{iT} \delta \mathbf{w}_i'(l_i) = 0 \end{aligned} \quad (10)$$

where $d(\cdot)/dt$ denotes the total time derivative of (\cdot) regarded as a function of the independent variables t , x_i , $\mathbf{w}_i(x_i, t)$, $\dot{\mathbf{w}}_i(x_i, t)$, $\mathbf{w}_i'(x_i, t)$, $\dot{\mathbf{w}}_i'(x_i, t)$, $\mathbf{w}_i''(x_i, t)$, $\dot{\mathbf{w}}_i''(x_i, t)$, $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t)$, and $\mathbf{w}_i(l_i, t)$, $\dot{\mathbf{w}}_i(l_i, t)$. In addition, each Lagrangian function is defined as follows:

$$\hat{L}_B = L_B + \sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{L}^i \, dx_i, \quad L = L_D + \sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{L}^i \, dx_i + L_B \quad (11)$$

The previous Lagrange's equations apply to general multiply connected hybrid coordinate systems for which discrete boundary elements are also included. These equations are a generalization of developments¹⁸ for a single flexible member. The boundary conditions described by Eq. (10) are associated with the governing equations at the boundary of each elastic domain. Motivated by the development of the hybrid version of Lagrange's equations, we seek to write the analogous hybrid version of Hamilton's canonical equations and to use those equations in deriving a stabilizing control law in a rather general form. More restricted versions of Hamilton's canonical equations for a continuous system are available in classical mechanics texts such as Ref. 19. These formulations, however, do not consider either hybrid coordinate systems or multibody systems with boundary terms in the Lagrangian.

As a generalization of the classical approach, the Hamiltonian of a hybrid coordinate system consisting of n one-dimensional domains can be written as

$$H = H_D + \sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{H}^i \, dx_i + H_B \quad (12)$$

where the Hamiltonian contributions corresponding to each substructure are

$$\begin{aligned} H_D &= \left(\frac{\partial L_D}{\partial \dot{\mathbf{q}}} \right)^T \dot{\mathbf{q}} - L_D(\mathbf{q}, \dot{\mathbf{q}}) \\ \hat{H}^i &= \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{q}}} \right)^T \dot{\mathbf{q}} + \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i} \right)^T \dot{\mathbf{w}}_i + \left(\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'} \right)^T \dot{\mathbf{w}}_i' \\ &\quad + \left[\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i(l)} \right]^T \dot{\mathbf{w}}_i(l) + \left[\frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'(l)} \right]^T \dot{\mathbf{w}}_i'(l) - \hat{L}^i \\ H_B &= \left(\frac{\partial L_B}{\partial \dot{\mathbf{q}}} \right)^T \dot{\mathbf{q}} + \left[\frac{\partial L_B}{\partial \dot{\mathbf{w}}(l)} \right]^T \dot{\mathbf{w}}(l) + \left[\frac{\partial L_B}{\partial \dot{\mathbf{w}}'(l)} \right]^T \dot{\mathbf{w}}'(l) - L_B \end{aligned} \quad (13)$$

As an extension to the case of discrete systems, we define generalized momentums of the system in conjunction with the structure of the Hamiltonian in Eq. (13). For each contribution to the Hamiltonian, the corresponding components of generalized momentum are defined as follows:

$$\begin{aligned} p_D &= \frac{\partial L_D}{\partial \dot{\mathbf{q}}}, \quad \hat{p}^i = \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{q}}}, \quad p_B = \frac{\partial L_B}{\partial \dot{\mathbf{q}}} \\ \Phi^i &= \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i}, \quad \Phi_s^i = \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'}, \quad \tilde{\Phi}^i = \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i(l)}, \quad \tilde{\Phi}_s^i = \frac{\partial \hat{L}^i}{\partial \dot{\mathbf{w}}_i'(l)} \\ \Phi_B &= \frac{\partial L_B}{\partial \dot{\mathbf{w}}(l)}, \quad \Phi_B^s = \frac{\partial L_B}{\partial \dot{\mathbf{w}}'(l)} \end{aligned} \quad (14)$$

In lieu of one vector of discrete conjugate momentum coordinates, we now have eight new subsets of conjugate momentum coordinates. Taking the differential of the Hamiltonian, we obtain

$$\begin{aligned} dH &= \dot{\mathbf{q}}^T d\mathbf{p}_D - \left(\frac{\partial L_D}{\partial \mathbf{q}} \right)^T d\mathbf{q} \\ &\quad + \sum_{i=1}^n \left(\int_{l_{0i}}^{l_i} \left\{ \dot{\mathbf{q}}^T d\hat{p}^i - d\mathbf{q}^T \left(\frac{\partial \hat{L}^i}{\partial \mathbf{q}} \right) + \dot{\mathbf{w}}_i^T d\Phi^i \right. \right. \\ &\quad \left. \left. - \left(\frac{\partial \hat{L}^i}{\partial \mathbf{w}_i} \right)^T d\mathbf{w}_i + \dot{\mathbf{w}}_i'^T d\Phi_s^i - \left(\frac{\partial \hat{L}^i}{\partial \mathbf{w}_i'} \right)^T d\mathbf{w}_i' \right. \right. \\ &\quad \left. \left. + \dot{\mathbf{w}}_i(l)^T d\tilde{\Phi}^i - \left[\frac{\partial \hat{L}^i}{\partial \mathbf{w}_i(l)} \right]^T d\mathbf{w}_i(l) + \dot{\mathbf{w}}_i'(l)^T d\tilde{\Phi}_s^i \right. \right. \\ &\quad \left. \left. - \left[\frac{\partial \hat{L}^i}{\partial \mathbf{w}_i'(l)} \right]^T d\mathbf{w}_i'(l) - \left(\frac{\partial \hat{L}^i}{\partial \mathbf{w}_i''} \right)^T d\mathbf{w}_i'' \right\} dx_i \right) \\ &\quad + \dot{\mathbf{q}}^T d\mathbf{p}_B - \left(\frac{\partial L_B}{\partial \mathbf{q}} \right)^T d\mathbf{q} + \dot{\mathbf{w}}(l)^T d\Phi_B - \left[\frac{\partial L_B}{\partial \mathbf{w}(l)} \right]^T d\mathbf{w}(l) \\ &\quad + \dot{\mathbf{w}}'(l)^T d\Phi_B^s - \left[\frac{\partial L_B}{\partial \mathbf{w}'(l)} \right]^T d\mathbf{w}'(l) \end{aligned} \quad (15)$$

On the other hand, the Hamiltonian of a hybrid system can be regarded as a function of generalized coordinates and the generalized momentums. Hence, apart from the previous expression, the Hamiltonian can be written functionally as

$$\begin{aligned} H &= H_D(\mathbf{q}, p_D) + \sum_{i=1}^n \int_{l_{0i}}^{l_i} \\ &\quad \times \hat{H}^i[\mathbf{q}, \hat{p}^i, \mathbf{w}_i, \mathbf{w}_i', \mathbf{w}_i'', \Phi_i, \Phi_s^i, \tilde{\Phi}^i, \tilde{\Phi}_s^i, \mathbf{w}(l), \mathbf{w}'(l)] \, dx_i \\ &\quad + H_B[\mathbf{w}(l), \mathbf{w}'(l), \Phi_B, \Phi_B^s, \mathbf{q}, p_B] \end{aligned} \quad (16)$$

Using the definition of the Hamiltonian in Eq. (16), we express the differential of the Hamiltonian in terms of differen-

tials of the generalized hybrid coordinates and generalized momentums to obtain

$$\begin{aligned}
 dH = & \left(\frac{\partial H}{\partial q} \right)^T dq + \left(\frac{\partial H}{\partial p_D} \right)^T dp_D + \sum_{i=1}^N \left(\int_{l_{0i}}^{l_i} \left\{ \left(\frac{\partial \hat{H}_i}{\partial q} \right)^T dq \right. \right. \\
 & + \left(\frac{\partial \hat{H}_i}{\partial \hat{p}^i} \right)^T d\hat{p}^i + \left(\frac{\partial \hat{H}_i}{\partial w_i} \right)^T dw_i + \left(\frac{\partial \hat{H}_i}{\partial w_i'} \right)^T dw_i' + \left(\frac{\partial \hat{H}_i}{\partial w_i''} \right)^T dw_i'' \\
 & + \left(\frac{\partial \hat{H}_i}{\partial \Phi^i} \right)^T d\Phi^i + \left(\frac{\partial \hat{H}_i}{\partial \Phi_s^i} \right)^T d\Phi_s^i + \left(\frac{\partial \hat{H}_i}{\partial \Phi^i} \right)^T d\Phi^i \\
 & + \left(\frac{\partial \hat{H}_i}{\partial \Phi_s^i} \right)^T d\Phi_s^i + \left[\frac{\partial \hat{H}_i}{\partial w(L)} \right]^T d\bar{w}(L) \\
 & + \left[\frac{\partial \hat{H}_i}{\partial w'(L)} \right]^T d\bar{w}'(L) \Big\} dx_i \Big) + \left[\frac{\partial H_B}{\partial w(L)} \right]^T d\bar{w}(L) \\
 & + \left(\frac{\partial H_B}{\partial \Phi_B} \right)^T d\Phi_B + \left(\frac{\partial H_B}{\partial q} \right)^T dq + \left(\frac{\partial H_B}{\partial p_B} \right)^T dp_B \\
 & + \left[\frac{\partial H_B}{\partial w'(L)} \right]^T d\bar{w}'(L) + \left(\frac{\partial H_B}{\partial \Phi_B^s} \right)^T d\Phi_B^s \quad (17)
 \end{aligned}$$

Then both expressions in Eqs. (15) and (17) are compared, equating the corresponding terms to produce the hybrid version of Hamilton's canonical equations. Lagrange's equations from Eqs. (9) and (10) are used in this procedure in the same context as the analogous equations for discrete systems. The results are

$$\dot{q} = \frac{\partial H}{\partial p} \quad (18)$$

$$\frac{d}{dt} p = -\frac{\partial H}{\partial q} + Q \quad (19)$$

$$\frac{d}{dt} \Phi^i - \frac{d}{dt} \frac{\partial}{\partial x_i} \Phi_s^i = \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{H}^i}{\partial w_i'} \right) - \frac{\partial \hat{H}^i}{\partial w_i} - \frac{\partial^2}{\partial x_i^2} \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right) + \hat{f}^i \quad (20)$$

and the boundary conditions are also expressed in terms of the Hamiltonian as

$$\begin{aligned}
 & \left[-\frac{d}{dt} \Phi_s^i - \frac{\partial \hat{H}^i}{\partial w_i'} + \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right) \right]^T \delta w_i \Big|_{l_{0i}}^{l_i} \\
 & + \left[-\frac{\partial H_B}{\partial w_i(l_i)} - \frac{d}{dt} \Phi_i \right]^T \delta w_i(l_i) + f_1^T \delta w_i(l_i) = 0 \quad (21) \\
 & - \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right)^T \delta w_i' \Big|_{l_{0i}}^{l_i} - \left[\frac{\partial H_B}{\partial w_i'(l_i)} + \frac{d}{dt} \Phi_i \right]^T \delta w_i'(l_i) \\
 & + f_2^T \delta w_i'(l_i) = 0
 \end{aligned}$$

where $(\cdot)^i$ denotes states associated with the i th domain or the i th boundary elements, and $(\cdot)_i$ represents i th component of (\cdot) . In addition,

$$\begin{aligned}
 H = H_D + \sum_{i=1}^N \int_{l_{0i}}^{l_i} \hat{H}_i dx_i + H_B, \quad \underline{H}_B = H_B + \sum_{i=1}^N \int_{l_{0i}}^{l_i} \hat{H}_i dx_i \quad (22) \\
 \hat{\Phi}_i = \left(\Phi_B + \sum_{j=1}^n \Phi_j^i \right)_i, \quad \hat{\Phi}_i = \left(\Phi_B^s + \sum_{j=1}^n \Phi_j^s \right)_i
 \end{aligned}$$

The generalized momentum vector p associated with the discrete generalized coordinate system q is

$$p = p_D + \sum_{i=1}^N \int_{l_{0i}}^{l_i} \hat{p}^i dx_i + p_B \quad (23)$$

So far, we have derived the Hamilton's canonical equations for a class of multiply connected hybrid coordinate systems.

As noted earlier, the Hamiltonian of a natural system is the total system energy and is often an attractive Lyapunov function candidate for establishing stabilizing control laws. Some of the existing Lyapunov approaches¹¹⁻¹⁵ for flexible structures use the governing equations of motion to eliminate acceleration coordinates that appear in the Lyapunov function time derivative, and one must carry out a large volume of algebra and calculus in most applications to multibody problems. A primary objective here is to develop a general form for stabilizing control laws based on the kinematic work rate of the control forces that holds irrespective of constitutive assumptions on the governing equations of motion.

B. Globally Stabilizing Control Law for Hybrid Coordinate Systems

We restrict our attention to natural hybrid coordinate systems to establish a Lyapunov function for control design so that we can most easily justify modifying the Hamiltonian of the system. Hence, we choose a positive definite Lyapunov function U in the form

$$U = H + af(e_q) = H_D + \sum_{i=1}^N \int_{l_{0i}}^{l_i} \hat{H}^i dx_i + H_B + af(e_q) \quad (24)$$

where $a > 0$ and $e_q \equiv q - q_f$ is an error vector with respect to a constant target state q_f . Assuming that our desired maneuver for a flexible structure is a near-rigid-body motion with simultaneous vibration suppression of flexible parts, the *final target state* is given by the rigid-body state

$$(q, \dot{q}, w_i, \dot{w}_i, w_i', \dot{w}_i', w_i'', \dot{w}_i'') = (q_f, 0, 0, 0, 0, 0, 0, 0) \quad (25)$$

In addition, we admit for generality the function $f(e_q) > 0$ as a pseudopotential energy consisting of error energy of discrete coordinate systems. If certain coordinates are cyclic, the $f(e_q)$ can be chosen to render an otherwise positive semidefinite U a positive definite function whose global minimum occurs at the desired state. Taking the time derivative of U yields

$$\begin{aligned}
 \frac{dU}{dt} = & \left(\frac{\partial H}{\partial q} \right)^T \dot{q} + \left(\frac{\partial H}{\partial p} \right)^T \dot{p} + \sum_{i=1}^N \left(\int_{l_{0i}}^{l_i} \left\{ \left(\frac{\partial \hat{H}^i}{\partial w_i} \right)^T \dot{w}_i \right. \right. \\
 & + \left(\frac{\partial \hat{H}^i}{\partial w_i'} \right)^T \dot{w}_i' + \left[\frac{\partial \hat{H}^i}{\partial w(L)} \right]^T \dot{\bar{w}}(L) + \left[\frac{\partial \hat{H}^i}{\partial w'(L)} \right]^T \dot{\bar{w}}'(L) \\
 & + \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right)^T \dot{w}_i'' + \dot{w}_i^T \frac{d}{dt} \Phi^i + \dot{w}_i'^T \frac{d}{dt} \Phi_s^i \\
 & + \dot{w}_i''^T \frac{d}{dt} \Phi^i + \dot{w}_i'^T \frac{d}{dt} \Phi_s^i \Big\} dx_i \Big) + \dot{w}(L)^T \Phi_B \\
 & + \dot{w}'(L)^T \Phi_B^s + \frac{d}{dt} \Phi_B^T \dot{\bar{w}}(L) + \dot{\bar{w}}'(L)^T \frac{d}{dt} \Phi_B^s + a \left[\frac{\partial f(e_q)}{\partial e_q} \right]^T \dot{q} \quad (26)
 \end{aligned}$$

Substitution of Hamilton's canonical equations in Eqs. (18-20) into Eq. (26) and integration by parts yield

$$\begin{aligned}
 \frac{dU}{dt} = & \dot{q}^T \left[Q + a \frac{\partial f(e_q)}{\partial e_q} \right] + \sum_{i=1}^N \left\{ \int_{l_{0i}}^{l_i} \dot{w}_i^T f^i dx_i \right. \\
 & + \left[\frac{\partial \hat{H}^i}{\partial w_i'} - \frac{\partial}{\partial x_i} \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right) + \frac{d}{dt} \Phi_s^i \right]^T \dot{w}_i \Big|_{l_{0i}}^{l_i} + \left(\frac{\partial \hat{H}^i}{\partial w_i''} \right)^T \dot{w}_i' \Big|_{l_{0i}}^{l_i} \Big\} \\
 & + \sum_{i=1}^N \left\{ \left[\frac{\partial H_B}{\partial w_i(l_i)} + \frac{d}{dt} \Phi_i \right]^T \dot{w}_i(l_i) \right. \\
 & + \left[\frac{\partial H_B}{\partial w_i'(l_i)} + \frac{d}{dt} \Phi_i \right]^T \dot{w}_i'(l_i) \Big\} \quad (27)
 \end{aligned}$$

Upon using the boundary conditions in Eq. (21), the previous expression is simplified to obtain the power equation

$$\begin{aligned} \frac{dU}{dt} = & \dot{q}^T \left[Q + a \frac{\partial f(e_q)}{\partial e_q} \right] \\ & + \sum_{i=1}^n \left[\int_{l_{0i}}^{l_i} \dot{w}_i^T \hat{f}^i dx_i + \dot{w}_i^T(l_i) f_1^i + \dot{w}_i'(l_i) f_2^i \right] \end{aligned} \quad (28)$$

The Lyapunov stability condition requires $dU/dt < 0$, and the simplest class of stabilizing control laws are constructed as the following feedback laws:

$$\begin{aligned} Q = & -K_1 \dot{q} - a \frac{\partial f(e_q)}{\partial e_q}, \quad \hat{f}^i(x_i, t) = -K_{2i} \dot{w}_i(x_i, t) \\ & f_1^i(l_i, t) = -K_{3i} \dot{w}_i(l_i), \quad f_2^i(l_i, t) = -K_{4i} \dot{w}_i'(l_i) \end{aligned} \quad (29)$$

where the gain matrices K_1 , K_{2i} , K_{3i} , and K_{4i} are positive definite matrices so that dU/dt becomes at least negative semidefinite. Of course, the control laws of Eq. (29) require that every term of Eq. (28) be dissipative. This is certainly sufficient, but it is not always necessary. The final form of the control laws in Eq. (29) are in output feedback form for which collocated sensor/actuator systems are implicitly assumed. Moreover, the stability guarantees of the control laws developed in this fashion are robust with respect to the truncation errors because no spatial discretization is involved in the process of establishing a stable family of control law designs. All kinds of modeling assumptions and approximations for establishing equations of motion (such as Coriolis effects and generalizations of the Euler-Bernoulli beam assumptions) can be absorbed into the Lyapunov function through the definition of the Hamiltonian. Thus, Eq. (28) holds for an extremely large family of constitutive and kinematic assumptions. The fundamental truth is that Eq. (28) is valid for a very large family of system models, guaranteeing that dU/dt is decreasing using Eq. (29) will lead to stability, if the actual system's U function is positive definite with its global minimum at the target state.

C. Globally Stabilizing Control Law Including Boundary Force Feedback

Motivated by the previous development, we develop a more general form of control laws that permits feedback on internal boundary forces between different substructures. In recent literature, boundary force feedback has been reported^{12-15,20} and its utility discussed for active vibration suppression. Motivated especially by Fujii's work,¹² we adopt a Lyapunov function as a weighted combination of substructure energy; the simplest such parameterization in the present discussion is

$$U = H_D + a_1 \left(\sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{H}_i dx_i + H_B \right) + a_2 f(e_q) \quad (30)$$

where $a_1 > 0$ and $a_2 > 0$. Note that the weighting constants $a_i > 0$ decide the relative importance of the distribution of energy between rigid and flexible parts of the structure during the maneuver. The weighting constant a_1 will be found to be related to the feedback gain on the boundary force between discrete elements and elastic structures; the value assigned to this weight turns out to have a significant effect on the closed-loop response of the system. Obviously many other physically motivated parameterizations of the system energy are possible and may prove attractive for specific applications. Note that Eq. (30) can be rewritten as

$$U = H + k_0 \left(\sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{H}_i dx_i + H_B \right) + a_2 f(e_q) \quad (31)$$

where $k_0 = a_1 - 1 > -1$, and time derivative of U is given by

$$\frac{dU}{dt} = \frac{dH}{dt} + k_0 \frac{d}{dt} \left(\sum_{i=1}^n \int_{l_{0i}}^{l_i} \hat{H}_i dx_i + H_B \right) + a_2 \dot{q}^T \frac{\partial f(e_q)}{\partial e_q} \quad (32)$$

Following the same procedure as the previous development, we obtain

$$\begin{aligned} \frac{d}{dt} \left(\int_{l_{0i}}^{l_i} \hat{H}_i dx_i + H_B \right) = & \dot{q}^T \left[\int_{l_{0i}}^{l_i} \left(\frac{d\hat{p}_i}{dt} + \frac{\partial \hat{H}^i}{\partial q} \right) dx_i + \frac{\partial H_B}{\partial q} \right. \\ & \left. + \frac{dp_B}{dt} \right] + \dot{w}_i^T(l_i) f_1^i + \dot{w}_i'(l_i) f_2^i + \int_{l_{0i}}^{l_i} \dot{w}_i^T \hat{f}^i dx_i \end{aligned} \quad (33)$$

Then substituting Eqs. (28) and (33) into Eq. (32), we obtain

$$\begin{aligned} \frac{dU}{dt} = & \dot{q}^T \left\{ Q + a_2 \frac{\partial f(e_q)}{\partial e_q} + k_0 \left[\sum_{i=1}^n \int_{l_{0i}}^{l_i} \left(\frac{d\hat{p}_i}{dt} + \frac{\partial \hat{H}^i}{\partial q} \right) dx_i \right. \right. \\ & \left. \left. + \frac{\partial H_B}{\partial q} + \frac{dp_B}{dt} \right] \right\} + a_1 \sum_{i=1}^n \left[\dot{w}_i^T(l_i) f_1^i + \dot{w}_i'(l_i) f_2^i \right. \\ & \left. + \int_{l_{0i}}^{l_i} \dot{w}_i^T \hat{f}^i dx_i \right] \end{aligned} \quad (34)$$

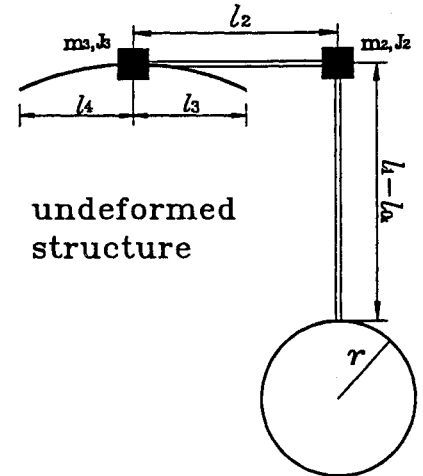


Fig. 2 Undeformed configuration of the structure.

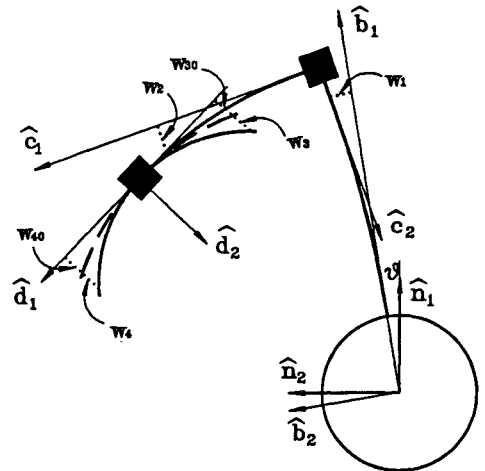


Fig. 3 Deformed configuration of the structure.

Table 1 Configuration parameters ($i = 1, 2, 3, 4$)

Parameter	Symbol	Value
Hub radius, in.	r	5.0
Rotary inertia of hub, oz-s ² -in.	I_h	50.0
Mass density of beams, oz-s ² /in. ²	ρ_i	0.003
Elastic modulus of the arms, oz/in. ²	E_i	161.6×10^6
Length of the first and second arms, in.	l_1, l_2	47.57, 10.0
Length of the third and fourth arms, in.	l_3, l_4	5.0, 5.0
Arm thickness, in.	t_i	0.118
Arm height, in.	h_i	5.94
First tip mass, oz-s ² /in.	m_2	0.3
Second tip mass, oz-s ² /in.	m_3	0.3
Rotary inertia of the tip masses, oz-s ² /in.	J_2, J_3	0.00018

Table 2 Natural frequencies

Mode no.	Frequency, Hz
1	0
2	1.65
3	3.19
4	10.35
5	26.72
6	50.43
7	60.42
8	88.65
9	113.15
10	152.06
11	175.44
12	204.26
13	258.05
14	330.59
15	411.22
16	504.93
17	566.17
18	667.89
19	741.74
20	817.10

Thus the simplest class of stabilizing control laws is

$$\begin{aligned}
 Q = & -k_0 \left[\sum_{i=1}^n \int_{l_{0i}}^{l_i} \left(\frac{d\hat{p}_i}{dt} + \frac{\partial \hat{H}^i}{\partial q} \right) dx_i + \frac{\partial H_B}{\partial q} + \frac{dp_B}{dt} \right] \\
 & - K_1 \dot{q} - a_2 \frac{\partial f(e_q)}{\partial e_q} \\
 \hat{f}^i(x_i, t) = & -K_{2i} \dot{w}_i(x_i, t) \\
 f_1^i(l_i, t) = & -K_{3i} \dot{w}_i(l_i), \quad f_2^i(l_i, t) = -K_{4i} \dot{w}_i'(l_i)
 \end{aligned} \quad (35)$$

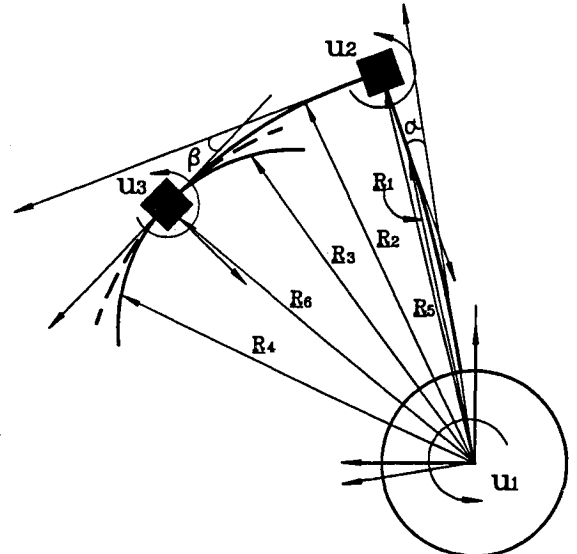
The control laws of Eq. (35) are in rather general forms, including feedback on boundary forces existing among the substructures. Note that any subset of the matrices $\{K_{2i}, K_{3i}, K_{4i}\}$ can be zeroed (thereby nullifying the necessity of measuring the associated elastic motions) without destabilizing the system. Once we know that Hamiltonian of a system, we can thus design a family of globally stabilizing control laws with a significant savings in the associated algebra for obtaining expressions for boundary force terms. In particular, for a nonlinear multiple flexible body system, the Hamiltonian approach provides a systematic way of synthesizing a guaranteed stabilizing structure for feedback control laws that avoid introducing the discretization and linearization modeling errors into the stability analysis. The control gains remain as free parameters with a rigorously guaranteed associated stable region of gain space. Optimization algorithms can now be introduced¹⁹ to obtain a best tuning of the control law over the stable region.

IV. Application to a Space Structure Model

An application of the control law design approach is now considered for a space structure model that consists of four elastic domains, two boundary elements, and one discrete element as in Figs. 2 and 3. This particular model is specifically motivated by the hardware experimental configuration by Hailey et al.²¹ Each elastic domain consists of a uniform beam element being connected through two discrete mass elements. The desired maneuver is a slew maneuver wherein an overall large angular motion is carried out with vibration suppression of the flexible appendages. The only rigid-body degree of freedom is the angular rotation θ of the hub, and this rigid-body motion introduces nonlinear coupling effects with other substructures (although small flexure is assumed, all other nonlinear kinematics are retained). Three control torque actuators are assumed to be available as in Fig. 4: one (u_1) at the rigid hub and the other two torques (u_2 and u_3) at the discrete mass elements.

The model configuration parameters are given in Table 1. Furthermore, the first 20 natural frequencies based on the linearized model configuration are listed in Table 2. Equations of motion of the model are quite lengthy and omitted here for brevity. The position vectors for the four flexible appendages and two discrete mass elements can be represented as in Fig. 4 with respect to the body-fixed axes δ_1 and δ_2 as follows:

$$\begin{aligned}
 \underline{R}_1 &= x_1 \delta_1 + w_1 \delta_2 \\
 \underline{R}_2 &= (l_1 - x_2 \sin \alpha - w_2 \cos \alpha) \delta_1 \\
 &\quad + [w_1(l_1) + x_2 \cos \alpha - w_2 \sin \alpha] \delta_2 \\
 \underline{R}_3 &= [l_1 - l_2 \sin \alpha - w_2(l_2) \cos \alpha - x_2 \cos(\alpha + \beta) \\
 &\quad - w_3 \cos(\alpha + \beta)] \delta_1 + [w_1(l_1) + l_2 \cos \alpha - w_2(l_2) \sin \alpha \\
 &\quad + x_3 \cos(\alpha + \beta) - w_3 \sin(\alpha + \beta)] \delta_2 \\
 \underline{R}_4 &= [l_1 - l_2 \sin \alpha - w_2(l_2) \cos \alpha + x_4 \sin(\alpha + \beta) \\
 &\quad - w_4 \cos(\alpha + \beta)] \delta_1 + [w_1(l_1) + l_2 \cos \alpha - w_2(l_2) \sin \alpha \\
 &\quad - x_4 \cos(\alpha + \beta) - w_4 \sin(\alpha + \beta)] \delta_2 \\
 \underline{R}_5 &= l_1 \delta_1 + w_1(l_1) \delta_2 \\
 \underline{R}_6 &= [l_1 - l_2 \sin \alpha - w_2(l_2) \cos \alpha] \delta_1 \\
 &\quad + [w_1(l_1) + l_2 \cos \alpha - w_2(l_2) \sin \alpha] \delta_2
 \end{aligned} \quad (36)$$

**Fig. 4** Position vectors and applied torques.

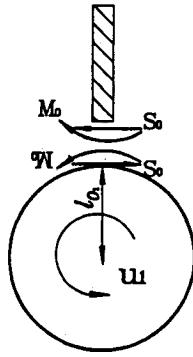
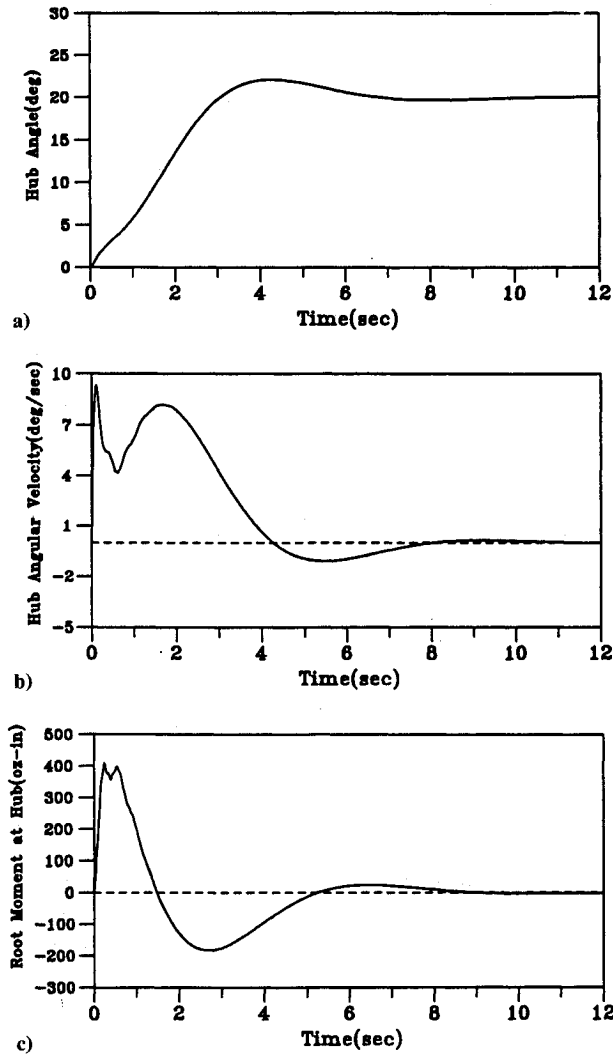


Fig. 5 Representation of internal boundary forces.

Fig. 6 Time response of the rigid hub: a) angle θ , b) angular velocity $\dot{\theta}$, and c) root moment $l_{01}S_0 - M_0$.

where α and β denote slopes of deflections at the tips of the first and second elastic domains, respectively, as presented in Fig. 4:

$$\alpha = \left. \frac{\partial w_1}{\partial x_1} \right|_{l_1}, \quad \beta = \left. \frac{\partial w_2}{\partial x_2} \right|_{l_2}$$

and where velocity vectors with respect to the inertial frame denoted by N of the position vectors are given by

$$\frac{N dR_i}{dt} = \frac{b dR_i}{dt} + N \underline{\omega}^b \times R_i, \quad N \underline{\omega}^b = \dot{\theta} \hat{b}_3 \quad (37)$$

For convenience of notation, we introduce

$$\frac{N dR_i}{dt} = v_i^1 \hat{b}_1 + v_i^2 \hat{b}_2 \quad (38)$$

where v_i^1 (v_i^2) is the velocity component of the i th substructure in the \hat{b}_1 (\hat{b}_2) direction. Then the Hamiltonian of the system is given in the form

$$H = H_D + \sum_{i=1}^4 \int_{l_{0i}}^{l_i} \hat{H}^i dx_i + H_B \quad (39)$$

where the velocity vectors of Eq. (38) are incorporated into

$$\int_{l_{0i}}^{l_i} \hat{H}^i dx_i = \frac{1}{2} \int_{l_{0i}}^{l_i} \left\{ \rho_i [(v_i^1)^2 + (v_i^2)^2] dx_i + E_i I_i \left(\frac{\partial \tilde{w}_i}{\partial x_i^2} \right)^2 \right\} dx_i$$

and

$$H_B = \frac{1}{2} m_2 [(v_5^1)^2 + (v_5^2)^2] + \frac{1}{2} m_3 [(v_6^1)^2 + (v_6^2)^2]$$

where $\tilde{w}_i \equiv w_i - w_{i0}$, and w_{i0} denotes static deformation of the i th elastic structure. The following Lyapunov function is proposed to design stabilizing control laws:

$$U = H_D + a_1 \left(\sum_{i=1}^4 \int_{l_{0i}}^{l_i} \hat{H}^i dx_i + H_B \right) + a_2 (\theta - \theta_f)^2 \quad (40)$$

where θ_f is a constant final target angle of the hub for a rest-to-rest maneuver of the whole structure. On the other hand, the nonconservative forces are given through the expression of virtual work as follows:

$$\delta W = (u_1 + u_2 + u_3) \delta \theta + (u_2 + u_3) \delta \alpha + u_3 \delta \beta \quad (41)$$

From Eq. (35) the simplest stabilizing control laws are directly obtained as

$$\begin{aligned} u_1 + u_2 + u_3 = & - (a_1 - 1) \left[\sum_{i=1}^4 \int_{l_{0i}}^{l_i} \left(\frac{d}{dt} \hat{p}^i + \frac{\partial \hat{H}^i}{\partial \theta} \right) dx_i \right. \\ & \left. + \frac{\partial H_B}{\partial \theta} + \frac{d}{dt} p_B \right] - k_1 \dot{\theta} - k_2 (\theta - \theta_f) \\ u_2 = u_3 = & -k_3 \dot{\alpha}, \quad u_3 = -k_4 \dot{\beta} \end{aligned} \quad (42)$$

Note that the bracketed right-hand side term of the first equation in Eq. (42), regarding θ as a discrete generalized coordinate, can be rewritten as

$$\begin{aligned} & \sum_{i=1}^4 \left[\int_{l_{0i}}^{l_i} \left(\frac{d}{dt} \hat{p}^i + \frac{\partial \hat{H}^i}{\partial \theta} \right) dx_i + \frac{\partial H_B}{\partial \theta} + \frac{d}{dt} p_B \right] \\ & = \sum_{i=1}^4 \int_{l_{0i}}^{l_i} \rho_i \frac{d}{dt} (v_i^1 c_i^1 + v_i^2 c_i^2) dx_i + m_2 \frac{d}{dt} (v_5^1 c_5^1 + v_5^2 c_5^2) \\ & \quad + m_3 \frac{d}{dt} (v_6^1 c_6^1 + v_6^2 c_6^2) \end{aligned} \quad (43)$$

where the parameters c_i^1 and c_i^2 ($i=1, 2, 3, 4$) are nonlinear functions of the system coordinates. On the other hand, the equation of motion for θ (which represents moment equilibrium over the whole structure) can be written as¹³⁻¹⁵

$$I_h \ddot{\theta} = - (l_{01} S_0 - M_0) + u_1 + u_2 + u_3 \quad (44)$$

$$\begin{aligned} l_{01} S_0 - M_0 = & \sum_{i=1}^4 \int_{l_{0i}}^{l_i} \rho_i \frac{d}{dt} (v_i^1 c_i^1 + v_i^2 c_i^2) dx_i \\ & + m_2 \frac{d}{dt} (v_5^1 c_5^1 + v_5^2 c_5^2) + m_3 \frac{d}{dt} (v_6^1 c_6^1 + v_6^2 c_6^2) \end{aligned}$$

where I_h is the rotary moment of inertia of the hub and M_0 and S_0 are internal boundary forces at the hub as in Fig. 5:

$$M_0 = E_1 I_1 \left. \frac{\partial^2 w_1}{\partial x_1^2} \right|_{l_{01}}, \quad S_0 = E_1 I_1 \left. \frac{\partial^3 w_1}{\partial x_1^3} \right|_{l_{01}} \quad (45)$$

Then the control laws in Eq. (42) can be rewritten as

$$\begin{aligned} u_1 + u_2 + u_3 &= -k_0(l_{01}S_0 - M_0) - k_1\dot{\theta} - k_2(\theta - \theta_f) \\ u_2 + u_3 &= -k_3\dot{\alpha}, \quad u_3 = -k_4\dot{\beta} \end{aligned} \quad (46)$$

where $k_0 = a_1 - 1 > -1$. Thus we find that the structure of a globally stabilizing control law for a multidomain hybrid coordinate system can be derived without going through complicated algebra. All of the system nonlinearities are absorbed into the control law designed through the introduction of the Hamiltonian of the system. The practical implementation of the previous control laws, including measurement of S_0 and M_0 , is discussed in Refs. 12–15.

V. Simulation

For simulation purposes, we discretized the original PDE system into a finite dimensional ODE system. The finite element method (FEM) is adopted by introducing the expansion

$$w_i(x, t) = \sum_{j=1}^4 \phi_{ji}^e(x_i) v_{ji}^e(t), \quad i = 1, 2, 3, 4 \quad (47)$$

where ϕ_{ji}^e is the j th shape (specified) function and v_{ji}^e is the j th nodal displacement associated with the i th elastic domain. In

fact, v_{1i}^e and v_{2i}^e (v_{3i}^e and v_{4i}^e) are transverse deflection and rotation at the left (right) nodal points of each finite element²² over the i th structure. By applying a standard FEM procedure with Eq. (47), the following linearized second-order system is obtained:

$$M\ddot{z} + Kz = Du \quad (48)$$

where z denotes a vector of combinations of rigid-body hub rotation and elastic degrees of freedom that are transverse deflection and rotation of the nodal point of each finite element of the corresponding elastic domain. In addition, M and K are mass and stiffness matrices, respectively. Even if the system under consideration is linear, however, the control laws stabilize a large class of systems, including nonlinear systems. As an example, the following data are used for feedback gains appearing in Eq. (46):

$$k_0 = -0.7, \quad k_1 = 600, \quad k_2 = 800, \quad k_3 = k_4 = 300$$

Simulation results for the system in Eq. (48) applied by the control torques in Eq. (47) are presented in Figs. 6–9. As it can be shown, the control laws stabilize the overall structure consisting of multi-elastic domains toward a constant target point for a rest-to-rest maneuver. Figure 6 shows rigid hub response where the hub angle and angular velocity are in highly damped motion. In addition to the two feedback gains k_1 and k_2 , the feedback gain k_0 on boundary force has significant effect on the closed-loop system.²⁰ The relatively large control torque (u_1) and root moment at the hub are also due to the bound-

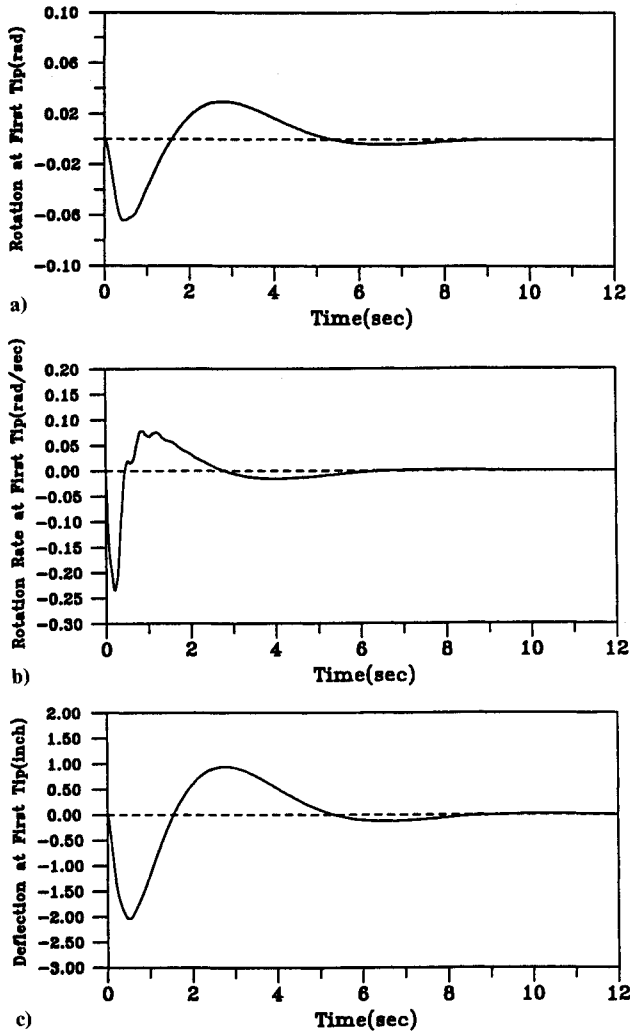


Fig. 7 Time response of the tip of first elastic domain: a) tip rotation α , b) rotation rate $\dot{\alpha}$, and c) tip deflection $w_1(l_1)$.

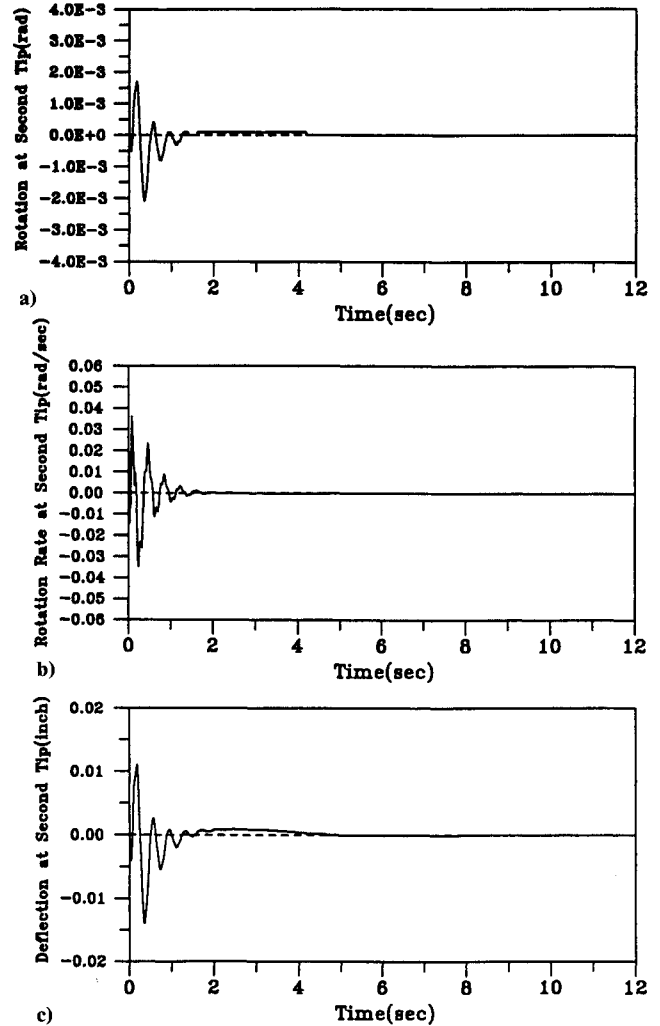


Fig. 8 Time response of the tip of second elastic domain: a) tip rotation β , b) rotation rate $\dot{\beta}$, and c) tip deflection $w_2(l_2)$.

ary force feedback. The effect of boundary force feedback in relation to its utility in vibration suppression, as well as sensitivity in actual implementation, needs further study. In Figs. 7 and 8, the responses at the two tips of elastic domains are presented. In this particular simulation, it is evident that the higher frequency modes play a significant role in the tip dynamics of the second elastic domain. Also, larger slope and deflections are observed at the first tip, as one might expect, due to the larger length of the first elastic domain. Applied control torques presented in Fig. 9 naturally correlate with the response of the points where the control torques are applied due to the nature of output feedback. The controller at second tip mass might require or benefit from a wider bandwidth actuator than the one at first tip mass due to the dominance of the higher modal frequencies in the tip rotational velocity; however, this one simulation provides insufficient evidence on this point. The magnitude of the control torque at the hub is almost the same order as root moment because of the feedback gain (k_0) on the root moment, which is chosen to be -0.7 in this simulation. Obviously, these closed-loop characteristics can be tuned by gain optimization over the stable region as was done in Ref. 20.

VI. Conclusions

A generalized Lyapunov approach for flexible hybrid discrete/distributed parameter systems has been established. The resulting Lyapunov stable control laws for nonlinear flexible systems are generated as functions of the system Hamiltonian. The boundary force feedback needs further study, should be

considered seriously for active vibration suppression with smaller settling time, and appears especially attractive for substructure vibration isolation. The simulation results show that the Lyapunov approach stabilizes a model system that is multiply connected, implying difficulty of conventional approaches such as state feedback and open-loop time-optimal control. As an extension of our approach, it is anticipated that tracking control laws for nonlinear flexible systems using Hamilton's canonical equations can be developed.

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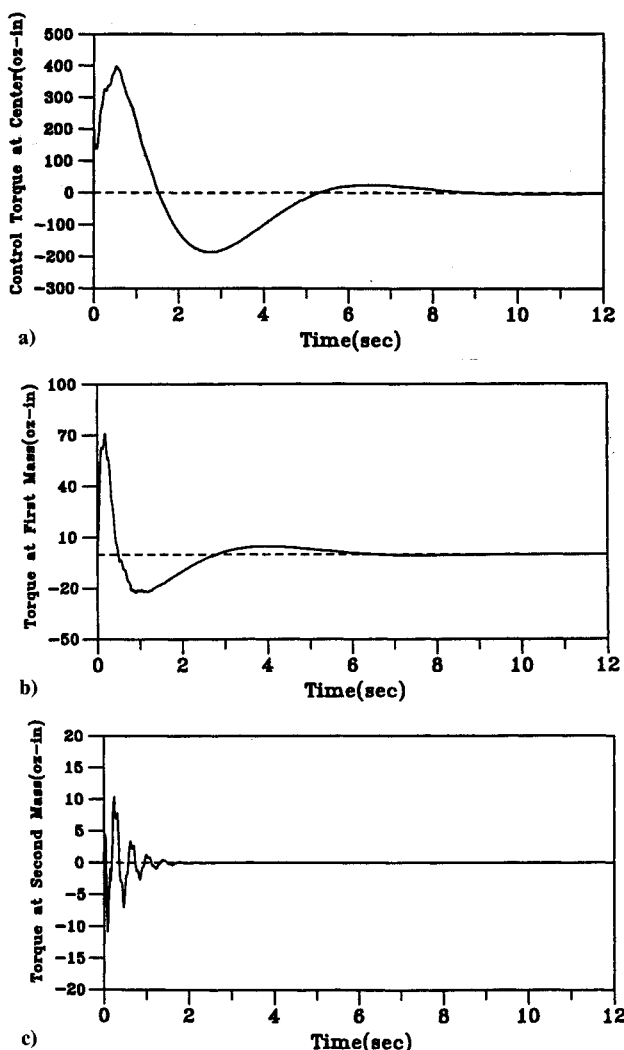


Fig. 9 Time response of applied torques at a) rigid hub u_1 , b) first mass u_2 , and c) second mass u_3 .